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(Received 5 May 2012; accepted 7 August 2012; published online 15 August 2012)

In this work first we review some cases where the action exhibits a minimal or a saddle-point criticality for velocity-independent potentials ($V(x, t)$) and maximum when the potential is velocity-dependent ($V(x, \dot{x}, t)$). In the following we will use the functional ("directional") derivative of second order to present a mathematically rigorous proof of the non-maximality of the classical functional action for potentials $V(x, t)$ velocity-independent. *Copyright 2012 Author(s). This article is distributed under a Creative Commons Attribution 3.0 Unported License*. [http://dx.doi.org/10.1063/1.4747508]

I. INTRODUCTION

The Hamilton principle is called by several authors the principle of least action.[1–4] Others refer to the stationary or critical action[5–7] but a majority of them do not discuss the nature of this criticality (maximum, saddle or minimum) because they do not follow it beyond the first-order functional variation, which is just enough to obtain the equations of motion of Euler-Lagrange and to proceed with their applications and dismemberments.

In 2007, C. Gray and E. Taylor[8] presented a discussion about *non-maximality* of stationary action and showed that the solution of the equation of the simple harmonic oscillator is *a saddle point of the action* by assuming that the time in the Lagrangian integral is greater than the semi-period of the oscillator, which is physically relevant to allow oscillations in this range. This is also contained in Chapter 6 of the book of David Morin.[9] In Ref. 10 we present a variety of models of *least action* for velocity-independent potentials and models of maximum action for velocity-dependent potentials.

In addition, Gray, Taylor and Morin have also shown an "intuitive proof" (expression contained in Ref. 8, which cite Refs. 9, 11, and 12) of a result which states that *the action can never be maximum if the potential of the Lagrangian is velocity-independent*, leaving for this potential, only the case of minimum or saddle. These demonstrations constitute one of the few references that discuss the subject and also have an important pedagogical character: an intuitive proof is equivalent to a plausible or reasonable argument, although not mathematically rigorous.

However, in this article we will work with the functional derivative of second order in the same line of reasoning of the Ref. 10 and so we realize a "rigorous proof" of non-maximality of the action functional for potentials $V(x, t)$ velocity-independent (A study to determine whether the non-maximality of the action of a given system is minimum or saddle was made by Gray and Taylor in Ref. 8 and involves the lenghts of the world lines and, in some cases, higher order derivatives.).

II. CALCULUS OF VARIATIONS AND MECHANICS

We consider the vector space $\mathcal{V} = C^2([0, T]; \mathbb{R})$ of functions $x : [0, T] \to \mathbb{R}$ twice continuously differenciable (laterally in the extrems 0 and $T < \infty$) and $\mathcal{F} \subset \mathcal{V}$,

$$\mathcal{F} = \{x \in \mathcal{V} \ | \ x(0) = x_0, \ \ x(T) = x_T \ \text{ fixed in } \ \mathbb{R}\}.$$

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Let be the functional $A : \mathcal{F} \to \mathbb{R}$ defined, for each $x \in \mathcal{F}$, by

$$A[x] = \int_0^T L(x(t), \dot{x}(t), t)dt \tag{1}$$

where $L : \mathcal{D} \times [0, T] \to \mathbb{R}, \mathcal{D} \subset \mathbb{R}^2$, is a C^2 function (e.g. the lagrangian of a particle moving in the X-axis) and $\dot{x}(t)$ is the derivative of x in t. Then following the same line of reasoning of the Ref. 10 (see also[13–16]), we have the Taylor expansion

$$A[x + \xi] = A[x] + \frac{1}{1!}D_\xi A[x] + \frac{1}{2!}D^2_\xi A[x] + E[\xi] \tag{2}$$

with $\xi \in \mathcal{V}$, not identically null, such that $\xi(0) = \xi(T) = 0$ (because $x + \xi \in \mathcal{F}$ for the expression $A[x + \xi]$ makes sense; we say that ξ is admissible) and

$$\lim_{\xi \mapsto 0} \frac{E[\xi]}{||\xi||^2} = 0. \tag{3}$$

Some remarks:

- $\mathcal{V}$ is normed space with norm $||\cdot||$ induced of the inner product $\langle x, y\rangle = \int_{[0,T]} x(t)y(t)dt$ defined for each $(x, y) \in \mathcal{V}^2$: $||\xi|| = \sqrt{\langle \xi, \xi\rangle}$ (it also has the induced metric of the norm: $d(x, y) = ||x - y||$).
- The directional (functional) derivative $D_\xi A[x]$ is defined by (see e.g. in the appendix of[17])

$$D_\xi A[x] = \frac{d}{d\epsilon}\{A[x + \epsilon\xi]\}\bigg|_{\epsilon=0} =$$

$$= \frac{d}{d\epsilon}\left[\int_0^T L(x(t) + \epsilon\xi(t), \dot{x}(t) + \epsilon\dot{\xi}(t), t)dt\right]_{\epsilon=0} \tag{4}$$

and, by derivation under integration simbol, using the chain rule and integrating by parts (with boundary conditions $\xi(0) = \xi(T) = 0$), we have

$$D_\xi A[x] = \int_0^T \left(\frac{\partial L}{\partial x}\xi + \frac{\partial L}{\partial \dot{x}}\dot{\xi}\right) dt$$

$$= \int_0^T \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\xi dt. \tag{5}$$

- When A is the action of the particle and x satisfies the Hamilton principle of the stationary (or critical) action,[9] that is $D_\xi A[x] = 0$ for all ξ admissible, then we have the equation of motion of Euler-Lagrange

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0. \tag{6}$$

- The directional (functional) derivative of the second order, $D^2_\xi A[x]$ is

$$D^2_\xi A[x] = D_\xi\{D_\xi A[x]\} = \frac{d}{d\epsilon}\{D_\xi A[x + \epsilon\xi]\}\bigg|_{\epsilon=0} \tag{7}$$

and then

$$D^2_\xi A[x] = \int_0^T \left(\frac{\partial^2 L}{\partial x^2}\xi^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\xi\dot{\xi} + \frac{\partial^2 L}{\partial \dot{x}^2}\dot{\xi}^2\right) dt \tag{8}$$

and so on for functional derivative of high order.

- For the *classical Lagrangian* with velocity-independent potential, $L = m\dot{x}^2/2 + V(x,t)$, we have

$$D_\xi^2 A[x] = \int_0^T \left[-\frac{\partial^2 V}{\partial x^2}\xi^2 + m\dot{\xi}^2 \right] dt. \tag{9}$$

- If $D_\xi A[x] = 0 \ \forall \ \xi$ admissible (Euler-Lagrange) then the Eq. (2) takes the form

$$A[x+\xi] = A[x] + \frac{1}{2!} D_\xi^2 A[x] + E(\xi), \quad \text{with} \quad \lim_{\xi \mapsto 0} \frac{E[\xi]}{||\xi||^2} = 0, \tag{10}$$

such that to determine if x is saddle or extremum local of A we can analize, whenever possible, the behavior of the second derivative $D_\xi^2 A[x]$.
- We say that $x \in \mathcal{F}$ is a point of maximum local of $A : \mathcal{F} \to \mathbb{R}$ if there is a neighborhood of x, we say $B_r(x) = \{y \in \mathcal{F}; \ \ ||y - x|| < r\}$ (centered on x and of radius r), such that $A[x] \geq A[x + \xi]$ for all $x + \xi \in B_r(x)$; analogously for x to be of minimum local, $A[x] \geq A[x + \xi]$ is replaced by $A[x] \leq A[x + \xi]$; if x is not point of maximum neither minimum, it is a saddle point of A (under the boundary conditions $x(0) = x(T) = 0$ a solution x of the Euler-Lagrange equation may be the unique, and therefore the "*global* critical point" of A, by theorem of existence and uniqueness for ordinary differential equation). Sometimes is convenient write, with ξ not identically null,

$$\xi = ||\xi||\eta \quad \text{where} \quad \eta = \frac{\xi}{||\xi||} \quad \text{is unitary (normalized).}$$

and by using Eq. (8), considering that $||\xi||(= \langle \xi, \xi \rangle^{1/2})$ is constant, we say equal to ε, we have $D_\xi^2 A[x] = ||\xi||^2 D_\eta^2 A[x] = \varepsilon^2 D_\eta^2 A[x]$ and then the signs of $D_\xi^2 A[x]$ and $D_\eta^2 A[x]$ are the same and, furthermore, we have

$$A[x+\xi] = A[x] + \frac{1}{2!}||\xi||^2 \left\{ D_\eta^2 A[x] + \frac{E[\xi]}{||\xi||^2} \right\}, \tag{11}$$

that is,

$$A[x+\varepsilon\eta] = A[x] + \frac{1}{2!}\varepsilon^2 \left\{ D_\eta^2 A[x] + \frac{\mathcal{R}[\varepsilon]}{\varepsilon^2} \right\}, \tag{12}$$

in which $E[\xi] = E[\varepsilon\eta] \equiv \mathcal{R}[\varepsilon] \mapsto 0$ more rapidly that ε^2, when $\varepsilon = ||\xi|| \mapsto 0$.

III. PROOF OF NON-MAXIMALITY OF THE ACTION FOR VELOCITY-INDEPENDENT POTENTIALS

Initially we present the following lemma.

Lemma: If x is point of maximum (respectively minimum) of A then $D_\xi^2 A[x]$ is non-positive (respectively non-negative), that is, $D_\xi^2 A[x] \leqslant 0$ (respectively $D_\xi^2 A[x] \geqslant 0$) for all admissible ξ.

Proof: We suppose that x is a point (actually, a function) maximizing A (the case for a minimum is analogous, Ref. 13). If $D_\xi^2 A[x] \leqslant 0 \ \forall \xi$ does not occur then $D_\xi^2 A[x] > 0$ for a ξ with $||\xi||$ sufficiently small and then from Eq. (10) or Eq. (11), it follows that $A[x + \xi] > A[x]$ since $E[\xi] \mapsto 0$ more rapidly that $||\xi||^2 \mapsto 0$ whenever $\xi \mapsto 0$ and then x is not point of maximum of A, contradition.

Considering the above lemma, we have

Theorem: Let be $\mathcal{F} = \{x \in C^2([0, T]; \mathbb{R}) | x(0) = x_0, \ \ x(T) = x_T\}$ and $A : \mathcal{F} \to \mathbb{R}$ defined by

$$A[x] = \int_0^T \left[\frac{1}{2} m\dot{x}^2 - V(x,t) \right] dt, \quad x \in \mathcal{F},$$

where $V : R \times [0, T] \to \mathbb{R}$, $R \subset \mathbb{R}$, is C^2. If $x \in \mathcal{F}$ is stationary point of A, in other words $D_\xi A[x] = 0$ for all $\xi \in C^2([0, T]; \mathbb{R})$ such that $\xi(0) = \xi(T) = 0$ (ξ admissible), that is if x satisfies Hamilton's principle, then x can not to maximize A.

Proof: If x to maximize A then, by lemma, $D^2_\xi A[x] \leqslant 0$ for all ξ admissible. Thence by (9)

$$\int_0^T \left[-\frac{\partial^2 V}{\partial x^2}\xi^2 + m\dot{\xi}^2 \right] dt \leqslant 0, \quad \forall \ \xi \ \text{ admissible.}$$

Therefore, there must exist a proper subset $I \equiv [a, b] \subset [0, T]$ such that $[\partial_x^2 V(x(t), t)][\xi(t)]^2 \geqslant m[\dot{\xi}(t)]^2$, $t \in I$. As $V(x(t), t)$ is C^2 in $I = [a, b]$, it follows from Weierstrass's theorem (Ref. 18, page 80), that exists $M \in \mathbb{R}$ $(M > 0)$ such that $\partial_x^2 V(x(t), t) \leqslant M$ for all $t \in I$; in particular for all $t \in I$ such that $\xi(t) \neq 0$ (ξ admissible) we have

$$m \left[\frac{\dot{\xi}(t)}{\xi(t)} \right]^2 \leqslant \left. \frac{\partial^2 V}{\partial x^2} \right|_{(x(t),t)} \leqslant M. \tag{13}$$

We choose $\xi(t) = t(T - t)(e^t - e^a)$ (C^2 and $\xi(0) = \xi(T) = 0$) and then

$$\lim_{t \mapsto a} \left[\frac{\dot{\xi}(t)}{\xi(t)} \right]^2 = \lim_{t \mapsto a} \left(\frac{1}{t} - \frac{1}{T - t} + \frac{e^a}{e^t - e^a} \right)^2 = +\infty$$

which contradicts (13). □

This theorem is also valid for n degrees of freedom ($n \in \mathbb{N} \equiv \{1, 2, 3, ...\}$). In this case the approach and the proof requires simple modifications. Now we consider functions $x : [0; T] \to \mathbb{R}^n$ (paths in $\mathbb{R}^n$) and the (C^2) Lagrangian $L : \mathcal{D} \times [0, T] \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^{2n}$,

$$L(x, \dot{x}, t) = \sum_{l=1}^{n} (m_l \dot{x}_l)/2 - V(x, t).$$

The Eq. (9) becames

$$D^2_\xi A[x] = \sum_{l=1}^{n} \int_0^T \left[-\frac{\partial^2 V}{\partial x_l^2}\xi_l^2 + m_l \dot{\xi}_l^2 \right] dt. \tag{14}$$

Is not difficult to see that if x maximizes A (reductio ad absurdum) we obtain, by similar arguments, an inequali-ty similar to (13):

$$m_l \left[\frac{\dot{\xi}_l(t)}{\xi_l(t)} \right]^2 \leqslant \left. \frac{\partial^2 V}{\partial x_l^2} \right|_{(x(t),t)} \leqslant M \tag{15}$$

for some $l \in 1, 2, ..., n$. The remainder of the proof is the same.

IV. CONCLUSION

As already commented in the introduction, the Ref. 10 by one of the authors presented examples where the action is maximum, minimum or saddle for velocity-dependent potentials. Gray, Taylor and Morin (Refs. already mentioned) showed how the action can be minimum or saddle for velocity-independent potentials. However, the intuitive proof presented by them for non-maximality of the action functional for velocity-independent potentials was "transformed" in our work into a (mathematically) rigorous proof by using the functional derivative of second order. Studies involving functional derivative of second order in the case of continuous systems and fields, with infinite *non-enumerable* degrees of freedom (Lagrangian densities, signature of the metric etc.), would merit a separate discussion.

ACKNOWLEDGMENTS

We thank Flávio Cruz (Universidade Regional do Cariri, Ceará, Brazil), Job Saraiva (Universidade Fede-ral de Alagoas, Maceió, Brazil), F. A. Brito (Universidade Federal de Campina Grande, Paraíba, Brazil) and Alan Costa (undergraduate student of the Universidade Regional do Cariri) for useful discussions. J.P.N.L. thanks FUNCAP-Ceará-Brazil for partial support.

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